In *Toward a Unified Theory of Problem Solving: Views From the Content Domain*, M. U. Smith, Ed., Lawrence Erlbaum Associates: Hillesdale, NJ, **1991**, pp.21-34

TOWARD A UNIFIED THEORY OF PROBLEM SOLVING: A VIEW FROM CHEMISTRY

by

George M. Bodner
Department of Chemistry
Purdue University
West Lafayette, IN 47907

Based on papers presented at the annual meeting of the American Educational Research Association (AERA), New Orleans, April, 1988 and the Tenth Annual Conference of the Cognitive Science Society, Montreal, August, 1988. Published in "Toward a Unified Theory of Problem Solving," M. U. Smith and V. L. Patel (Eds.), Lawrence Erlbaum Associates Publishers, 1990.

Introduction

For months, I have been agonizing over the question posed by the organizers of this symposium: "Is it possible to produce a unified theory of problem solving?" I have waffled back and forth between an optimistic "yes" and a pessimistic "no". While sitting in a hotel room, just before leaving for the 10th Annual Conference of the Cognitive Science Society, I came to the following conclusion.

> Yes, it is possible to construct a unified theory of problem solving. I have done so, and I expect that each of the other participants in this symposium will have done so as well. Unfortunately, I'm afraid our unified theories will differ significantly from one another.

I am confident that we are beyond the stage described by Figure 1, but I fear that there are subtle differences between the way each of us defines important terms, which cause difficulties in reaching a truly unified theory of problem solving. Researchers in this area, as much as any I've encountered, seem to adhere to a philosophy summarized by Lewis Carroll (1896).

> "When I use a word," Humpty Dumpty said in a rather scornful tone, "it means just what I choose it to mean, — neither more nor less."

We can't even agree among ourselves about the meaning of the word "problem", much less "problem solving". In his AERA paper, Mike Smith (1988) reported that he had fought with Don Woods about whether successful problem solvers most often used a forward-working versus means-ends analysis approach to problem solving. He then noted that the confusion was resolved when he realized that Woods does not consider the solution of exercises to be problem solving.

Smith (1988) argues cogently for including solving exercises as a subset of problem solving. To ensure debate, I'm going to disagree. I'm also going to question the notion that the difference between exercises and problems is one of “difficulty” or “complexity”. Finally, I'm going to question the assertion that individuals solving exercises use many of the same strategies they apply to problems. I don't question the validity of studying the solving of exercises, I just don't believe this is relevant to discussions of problem solving.

Definitions of Terms

Unlike many others, I do not feel compelled to introduce new definitions of terms. Let me simply state the operational set of definitions I will use in this paper. Hayes (1980) defined a problem as follows:

> Whenever there is a gap between where you are now and where you want to be, and you don't know how to find a way to cross that gap, you have a problem.

Wheatley (1984) coined the consummate definition of problem solving.

> What you do, when you don't know what to do.

Let me conclude this section by introducing a working definition of the term *algorithm* (Ehrlich et al., 1980).

> Rules for calculating something that can be followed more or less automatically by a reasonably intelligent system, such as a computer.

Logical Consequences of These Definitions

If you accept these definitions, there is a fundamental difference between an exercise and a problem. We all routinely encounter questions or tasks for which we don't know the answer, but we feel confident that we know how to obtain the answer. When this happens, when we know the sequence of steps needed to cross the gap between where we are and where we want to be, we are faced with an exercise not a problem.

Smith (1988) eliminates from classification as a problem any task that can be solved completely by an algorithm. I agree. By listening in class, by reading examples in the text, and, most importantly, by working similar questions on their own, most of my students construct an algorithm that turns the following question into an exercise.

> What is the empirical formula of a compound of xenon and oxygen that is 67.2% Xe and 32.8% O? (Carter, LaRussa and Bodner, 1987)

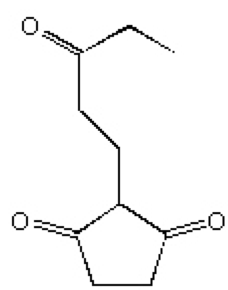
In fact, I would argue — in accord with Johnstone and El-Banna (1986) — that it is the existence of a well-defined algorithm, constructed from their prior experience (Bodner, 1986), that turns this question into an exercise for so many of my students.

There is no innate characteristic of a task that inevitably makes it a problem. Status as a problem is a subtle interaction between the task and the individual struggling to find an appropriate answer or solution (Bodner, 1987). The following question, for example, is a problem for most students when they begin their study of chemistry.

> Magnesium reacts with oxygen to form magnesium oxide.
>
> $$2\ Mg(s) + O_2(g) \rightarrow 2\ MgO(s)$$
>
> What weight of oxygen is required to burn 10.0 grams of magnesium?

To their instructors, however, it is an exercise. They have done so many similar calculations that they almost instantly recognize that they can convert grams of magnesium into moles of magnesium, moles of magnesium into moles of oxygen, and then moles of oxygen into grams of oxygen. Similarly, while the following task would be a problem for most readers of this chapter, it is no more than a routine exercise for chemists who specialize in organic synthesis.

> Robinson annulation reactions involve two steps: Michael addition and aldol condensation. Assume that Michael addition leads to the following intermediate. What would be produced when this intermediate undergoes aldol condensation?



The difference between an exercise and a problem is not a question of difficulty, or complexity, but one of familiarity. Patel and Groen (1988) noted that expert physicians explained routine medical diagnosis problems by a process of forward reasoning. They also noted that “forward reasoning is associated with successful performance and tends to disappear when a subject is uncertain or unsuccessful.” I'm not surprised. I would argue that the more likely that the protocol from a “problem-solving” interview can be analyzed as an example of forward-chaining or forward-working, the more likely the individual will be successful because he or she was working an exercise.

The Role of Algorithms in Working Exercises and Problems

Consider the algorithm used to solve the following question.

> What is the empirical formula of a compound of xenon and oxygen that is 67.2% Xe and 32.8% O?

Our students are taught to assume that they start with a 100-gram sample of the compound, which therefore contains 67.2 grams of xenon and 32.8 grams of oxygen. They are then taught to convert grams of each element into moles of the element.

$$67.2 \text{ g Xe} \times \frac{1 \text{ mol Xe}}{131.30 \text{ g}} = 0.512 \text{ mol Xe}$$

$$32.8 \text{ g O} \times \frac{1 \text{ mol O}}{15.9994 \text{ g}} = 2.05 \text{ mol O}$$

Finally, they are taught to divide the number of moles of xenon into the number of moles of oxygen to find the ratio of these elements

$$\frac{2.05 \text{ mol O}}{0.512 \text{ mol Xe}} = 4.00$$

and thereby conclude that the empirical formula of the compound is XeO_4. In essence, this is a five-step algorithm.

- Assume 100 grams of the compound.
- Convert percent-by-weight data into grams of each element.
- Calculate the number of moles of each element.
- Calculate the ratio of the moles of each element.
- Convert the answer to the simplest whole-number ratio, if necessary.

But what happens when we take out the hint that tells the students how to start this algorithm? Consider the following question.

> 9.33 g of copper metal was allowed to react with an excess of chlorine and it was found that 14.6 g of a compound of copper and chlorine were formed. What is the empirical formula of this compound? (Carter, LaRussa and Bodner, 1987)

If you apply the concept of M-demand (Pascual-Leone, 1970; Case, 1972; and Scardamalia, 1977) to this question, it has one or at most two more steps than the previous question. In theory, the known weight of copper (9.33 g) and chlorine (14.6 g - 9.33 g) in the sample could be converted into the percent by weight of these elements (63.9% Cu and 36.1% Cl). From that point, the algorithm for doing empirical formula calculations can be used intact.

But that isn't how either students or their instructors go about solving this problem. They seldom use the hint that this is an empirical formula question to construct a familiar problem (63.9% Cu and 36.1% Cl), which can be solved algorithmically. They use a totally different approach to the problem. Algorithms are still used, but they involve much smaller chunks of information. Instead of a five-step algorithm, they use simpler algorithms, which automate individual steps in the calculation, such as converting from grams to moles.

Archistic versus anarchistic approaches to problem solving

From the beginning of our work on problem solving in chemistry, it was apparent that there was little similarity between the strategies our expert problem solvers used to solve problems that were novel to them and the solutions to these problems presented in the instructor's manuals that accompanied the textbooks from which the problems were taken.

As Herron (1988) has described it,

> This difference between the problem-solving performance of experts and textbook solutions is significant because the examples must convey to the students an unrealistic idea about how problems are actually attacked. The examples provide no indication of the false starts, dead ends, and illogical attempts that characterize problem solving in its early stages, nor do they reveal the substantial time and effort expended to construct a useful representation of a problem before the systematic solution shown in examples is possible.

If you will allow me to coin a word — which I do reluctantly — I would like to describe these textbooks solutions as *archistic.*

The textbook solutions are perfectly logical sequences of steps that string together in a linear fashion from the initial information directly to the solution. They are perfect examples of how exercises would be worked by an individual who has many years of experience with similar tasks. They have little — if any — similarity, however, to the *anarchistic* approach experts use when they solve problems. (I owe a debt of gratitude to Lochhead (1979), who first described an anarchistic approach to *teaching* problem solving, which enabled me to recognize the role it plays in *doing* problem solving.)

Before I am accused of promoting the theory that all forms of government are oppressive and undesirable, let me remind you of another definition of anarchism: lacking order or control. In order to introduce an anarchistic model of problem solving, I must first comment on more established models.

Stage models for problem solving

Polya (1945) was the first, but by no means the last, to propose a model of problem solving that involves stages, such as:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

Our work has repeatedly shown the validity of this model for understanding how experts solve exercises. But it suggests that essentially all of the activities I would defined as problem solving occur during the first stage.

Consider the following question.

> A sample of a compound of xenon and fluorine was confined in a bulb with a pressure of 24 torr. Hydrogen was added to the bulb until the pressure was 96 torr. Passage of an electric spark through the mixture produced Xe and HF. After the HF was removed by reaction with solid KOH, the final pressure of xenon and unreacted hydrogen in the bulb was 48 torr. What is the empirical formula of the xenon fluoride in the original sample? (Holtzclaw, Robinson, and Nebergall, 1984)

If your knowledge of chemistry is rusty, let me remind you that the partial pressure of each gas in this question is directly proportional to the number of moles of gas particles in the sample. Try to solve this problem before you continue reading.

Did you use Polya's four-stage model? Did you start by constructing an understanding of the problem? Did you then devise a plan, or a sequence of steps for solving the problem, before you carried out the plan?

This question should be a problem for virtually everyone who reads this chapter. As you reflect on your experience with this question, I believe you will conclude that you didn't fully understand the problem until you solved it. Furthermore, I doubt that you went through a stage in which you designed a sequence of steps that would lead to the solution before you carried out these steps.

I believe that Polya's stage model of problem solving, and its numerous archistic descendants, are better models of what happens when people work exercises or familiar problems. It has little to do with what happens when they solve problems.

An anarchistic model of problem solving

Over the last four years, I have been refining a model of what expert problem solvers do when they work problems, which is based on a model first proposed by Grayson Wheatley (1984).

- Read the problem
- Now read the problem again
- Write down what you hope is the relevant information

- Draw a picture, make a list, or write an equation or formula to help you begin to understand the problem
- Try something
- Try something else
- See where this gets you
- Read the problem again
- Try something else
- See where this gets you
- Test intermediate results to see whether you are making any progress toward an answer
- Read the problem again
- When appropriate, strike your forehead and say, “Son of a ...”
- Write down *an* answer (not necessarily *the* answer)
- Test the answer to see if it makes sense
- Start over if you have to, celebrate if you don't

When this model was proposed at a seminar in the chemistry department at Purdue, one of my colleagues summarily rejected it. He argued that this is what we do when we do research, and stated that we can't expect students to approach problem solving the same way. I disagree.

The model of problem solving outlined above shares many of the characteristics that makes “science ... an essentially anarchistic enterprise” (Feyerabend, 1975). Whereas exercises are often worked in a linear, forward-chaining, rational manner, this model of problem solving is cyclic, reflective, and might even appear irrational to anyone watching us use it.

Importance of differentiating between exercises and problems

Why do I place so much emphasis on the difference between exercises and problems? In chemistry, we are already doing a fairly good job of teaching students to work exercises. We introduce them to certain classes of problems, such as the empirical formula calculations described earlier. We then lead them through enough similar questions until the successful ones build an algorithm for doing these calculations. As a result of this instruction, we have produced good exercise solvers. We are much less successful, however, at teaching them to be good problem solvers.

Some of my more pessimistic colleagues believe that the only realistic goal we can attain is have our students work more and more complex exercises. I pray they are wrong. I believe that the ultimate goal of the research in this area is to improve the problem-solving skills of our students.

Unfortunately, when my colleagues in chemistry read the problem-solving literature, they invariably walk away with models that suggest students should be able to work problems much the same way they work exercises. They therefore present beautiful algorithmic approaches to their students for working problems. When students fail at problem solving with these techniques, they conclude that their students are either stupid or lazy.

Ever since I began teaching, I have listened to colleagues bemoan the fact that beginning students seem to be able to handle one-step questions, or perhaps two-step questions, but not questions that are more complex. I think I understand why.

Several years ago, the students in my course were assigned the following question:

> A sample of indium bromide weighing 0.100 g reacts with silver nitrate, $AgNO_3$, giving indium nitrate and 0.159 g of AgBr. Determine the empirical formula of indium bromide. (Holtzclaw, et al., 1984)

This is a difficult question, which is a problem for all of my students and many of the teaching assistants as well. In virtually every recitation section, the students asked the TA to do this problem. I noticed that the essentially all of the TA's told their students that the problem could be worked more or less like this:

Start by converting grams of AgBr into moles of AgBr. Convert moles of AgBr into moles of Br and then convert moles of Br into grams of Br. Subtract grams of Br from grams of indium bromide to give grams of In. Convert grams of In into moles of In. Then divide moles of Br by moles of In to get the empirical formula of the compound.

During the next staff meeting, I asked the TA's to stop lying to the students. I told them that the technique for solving this problem they had presented to their students had little if anything to do with the process they had used to solve the problem for the first time. I argued that they had confused the process used to solve exercises with the process used to solve problems. Finally, I suggested that the description given above was an algorithm for solving similar questions, which they had constructed *after* they had solved this problem. Thus, it wasn't surprising when the students reported to me that they felt discouraged, because they weren't capable of solving the problem the way their TA's did.

What is the effect of not distinguishing between exercises and problems?

In a study of the role of beliefs in problem solving in chemistry, Carter (1987) reported her experience as a teaching assistant in a junior-level physical chemistry course. Her students found physical chemistry to be a difficult or frustrating obstacle on the way to engineering, chemistry, or biology degrees. She and her fellow TA's, on the other hand, were frustrated with the students. They felt that the students lacked basic problem-solving skills, in spite of their strong mathematical backgrounds. While they were adept at using algorithms and manipulating numbers, they seemed unable to apply their skills with basic chemical concepts when presented with a novel problem. Few seemed to see the need to develop complete, coherent representations of problems, and the equations they memorized seemed to be disconnected bits of information.

The main source of the TA's frustration, however, was that the students saw nothing wrong with this. Physical chemistry was supposed to be hard; it wasn't supposed to make sense. Because the students believed that they couldn't be expected to actually understand physical chemistry, they dealt with the material accordingly. They memorized equations, and worked examples of the same problem type over and over again, with little concern as to why that particular pattern or method worked, or why it did not.

The students' attempts to treat physical chemistry in terms of exercises, which can be handled by memorizing equations and algorithms for doing calculations, inevitably lead to frustration on the part of both the students and their instructors. The students were frustrated, because they weren't successful at this task. Their instructors were frustrated, because the students weren't behaving properly.

Implications of the anarchistic model of problem solving

The model proposed in this chapter has helped me understand many of the observations made during the 17 years I've taught general chemistry. I can understand why so many beginning students have difficulty learning how to do even the simplest stoichiometry calculations, such as the following:

> How much carbon dioxide is produced when 10.0 grams of sugar in the form of sucrose ($C_{12}H_{22}O_{11}$) react with excess oxygen?
>
> $$C_{12}H_{22}O_{11}(aq) + 12\ O_2(g) \rightarrow 12\ CO_2(g) + 11\ H_2O(l)$$

This task, which is a simple exercise for their instructors, is a problem to them. Until they stumble on the answer to enough problems of this nature to build their own algorithm for working them, watching their instructor do the calculation as an exercise isn't going to be sufficient. (An expert watching them approach these problems might consider their work to be “disorganized”, or even “irrational”, because it differed so much from the approach the expert would take. If the expert is the student's instructor, he or she might be tempted to intervene, to show the student the “correct” way of obtaining the answer. While this might make the instructor feel good, it doesn't necessarily help the student.)

This model has also helped me understand why so many students who successfully build an algorithm to do the calculation given above fail when they encounter the following limiting reagent question, which appears on the surface to be similar.

> How much carbon dioxide is produced when 10.0 grams of sugar in the form of sucrose ($C_{12}H_{22}O_{11}$) react with 10.0 grams of oxygen?
>
> $$C_{12}H_{22}O_{11}(aq) + 12\ O_2(g) \rightarrow 12\ CO_2(g) + 11\ H_2O(l)$$

In order to answer this question, one would have to calculate the amount of carbon dioxide that could be obtained from 10.0 grams of sucrose. But then one would have to recognize that there might not be enough oxygen to consume all of the sugar. One might then calculate the amount of oxygen that would be needed, and see if there is enough. Alternatively, one could calculate the amount of carbon dioxide that could be obtained from 10.0 grams of oxygen. Then, one would have to compare the results of these calculations, and decide which reagent is present in excess and which reagent limits the extent of reaction. Only then can the amount of CO_2 be predicted. In other words, there is more to solving this problem than applying stoichiometry algorithms in the correct order (Bodner and McMillen, 1986).

Implications of the anarchistic model for teaching

This model brings into question some of the techniques used to teach chemistry at present. As Herron and Greenbowe (1986) note:

> Virtually all problem-solving activities in standard courses focus on problems for which an algorithmic solution has been taught. ... Our research reveals that expert problem solvers never follow such direct paths when confronted with novel tasks, and most problems in chemistry are novel to the students in the course. We believe that we must give far more attention to how experienced problem solvers go about making sense out of problems encountered for the first time.

The model also provides a hint as to how to improve the teaching of chemistry. As Herron and Greenbowe note:

> Expert problem solvers make use of a number of general strategies (heuristics) as they interpret, represent, and solve problems. Trial and error, thinking of the problem in terms of the physical system discussed, solving a special case, solving a simple problem that seems related to a difficult problem, ... breaking the problem into parts, substituting numbers for variables, drawing diagrams to represent molecules and atoms, and checking interim or final results against other information ... are common strategies used by successful problem solvers. Although teachers frequently use these strategies, little attention is given to teaching these strategies to students.

Is it possible to improve the problem-solving performance of beginning chemistry students? Frank (1985) constructed a study to see if intervention during recitation sections could achieve this result. Students in the experimental section were encouraged to ask questions such as:

> What is the unknown? What are the conditions? What do these substances look like? How do the atoms and molecules involved here interact? How could you symbolize what you see?

Students in the experimental group outscored the control students on class exams. They formed more generalizable representations, were more persistent, and evaluated their work more frequently than control students. They also worked toward an inappropriate goal more often, perhaps as a result of increased confidence — these students worked toward an inappropriate goal under conditions where students in the control group either gave up or worked toward no goal at all.

When you consider that the intervention in Frank's study only occurred in recitation, these results are promising. They should be viewed with care, however. One of the instructor's in this study made the following observations.

> Some of my students seemed to enjoy group work and the emphasis on a few general principles. Others, however, were frustrated or threatened by the non-traditional approach. They believed chemistry teachers were supposed to tell them the answers or show them how to work problems. Since they were not told which equations to memorize, or "the" method of working each problem, they could not be expected to succeed on the exams. A conflict existed between my beliefs about the nature of learning chemistry and what my students believed ...; a conflict only compounded by the gap between my goal of improving their problem-solving skills and their goals, which often consisted of particular course grades. (Carter, 1987)

References

Bodner, G. M. (1986). Constructivism: A theory of knowledge. *Journal of Chemical Education, 63*, 873-878.

Bodner, G. M. and McMillen, T. L. B. (1986). Cognitive restructuring as an early stage in problem solving. *Journal of Research in Science Teaching, 23*, 727-737.

Bodner, G. M. (1987). The role of algorithms in teaching problem solving. *Journal of Chemical Education, 64*, 513-514.

Carroll, Lewis. (1896). *Through the looking glass*, Chapter 6.

Carter, C. S. (1987). The role of beliefs in general chemistry problem solving. Unpublished doctoral dissertation, Purdue University, West Lafayette, IN.

Carter, C. S., LaRussa, M. A., and Bodner, G. M. (1987). A study of two measures of spatial ability as predictors of success in different levels of general chemistry. *Journal of Research in Science Teaching, 24*, 645-657.

Case, R. (1972). Validation of a neo-Piagetian mental capacity construct. *Journal of Experimental Child Psychology, 14*, 287-302.

Ehrlich, E., Flexner, S. B., Carruth, G., and Hawkins, J. M. (1980). *Oxford American Dictionary*. Oxford: Oxford University Press.

Feyerabend, P. (1975). *Against method: Outline of an anarchistic theory of knowledge.* Thetford, Norfolk: Thetford Press Limited.

Frank, D. V. (1985). Implementing instruction to improve the problem-solving abilities of general chemistry students. Unpublished doctoral dissertation, Purdue University, West Lafayette, IN.

Hayes, J. (1980). *The complete problem solver*. Philadelphia: The Franklin Institute.

Herron, J. D. and Greenbowe, T. J. (1986). What can we do about Sue?: A case study of

competence. *Journal of Chemical Education, 63*, 528-531.
Herron, J. D. (1988). Research in chemical education: Results and directions. A paper presented at the conference: Toward A Scientific Practice of Science Education, Lawrence Hall of Science, Berkeley, CA.
Holtzlaw, H. F., Robinson, W. R., and Nebergall, W. H. (1984). *General Chemistry*, 7th Ed. Lexington: D. C. Heath
Johnstone, A. H. and El-Banna, H. (1986). Capacities, demands and processes — a predictive model for science education. *Education in Chemistry*, May, 80-84
Lochhead, J. (1979). An anarchistic approach to teaching problem solving methods. Paper presented at the annual meeting of the American Educational Research Association in San Francisco.
Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piaget's developmental stages. *Acta Psychologia, 32*, 301-345.
Patel, V. L. and Groen, G. (1988). Toward a unified theory of problem solving: a view from medicine. Paper presented at the annual meeting of the American Educational Research Association in New Orleans.
Polya, G. (1945). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.
Scardamalia, M. (1977). Information processing capacity and the problem of horizontal decalage: A demonstration using combination reasoning tasks. *Child Development, 48*, 28-37.
Smith, M. U. (1988). Toward a unified theory of problem solving: a view from biology. Paper presented at the annual meeting of the American Educational Research Association in New Orleans.
Wheatley, G. H. (1984). Problem solving in school mathematics. MEPS Technical Report 84.01, School Mathematics and Science Center, Purdue University, West Lafayette, IN.